Year 2023-2024

The Chinese University of Hong Kong

Department of Mathematics

MMAT 5340 Probability and Stochastic Analysis

Homework 1

Due Date: 23:59 pm on Tuesday, January 23, 2024.

Please submit your homework on Blackboard

Remark. *If you do not have the background in elementary probability theory, then the first three chapters of the textbook Probability, Statistics, and Stochastic Processes by Mikael Andersson and Peter Olofsson may be a good reference for you.*

1. Let X be a discrete random variable that has a binomial distribution with parameters n and p, written as $X \sim \text{Binomial}(n, p)$. Its probability mass function is given by

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \ldots$$

where $p \in (0, 1)$ is some constant. Compute the following values

a) $\mathbb{E}[X]$, $\mathbb{E}[X^2]$ and hence $\text{Var}[X]$.

b) $M_X(t) := \mathbb{E}[\exp(tX)]$, where $t \in \mathbb{R}$.

c) the derivatives at $t = 0$:

$$\frac{d}{dt} M_X(t)\bigg|_{t=0} \quad \text{and} \quad \frac{d^2}{dt^2} M_X(t)\bigg|_{t=0}.$$

These values should agree with the values of $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$ that you have obtained in part (a).

Hint 1: for a discrete random variable X, the expectation value of the random variable $g(X)$ is given by $\sum_x g(x) P(X = x)$. Here $g(X)$ is any function of X, for example, you may take $g(X) = X^2$.

Hint 2: You may find the the binomial theorem useful:

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

2. Let X be a continuous random variable that has a normal distribution with parameters μ and σ^2, written as $X \sim N(\mu, \sigma^2)$. Its probability density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R},$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are constants.

Compute the following values

a) $\mathbb{E}[X]$, $\mathbb{E}[X^2]$ and hence $\mathrm{Var}[X]$.

b) $M_X(t) := \mathbb{E}[\exp(tX)]$, where $t \in \mathbb{R}$.

c) the derivatives at $t = 0$:

$$\left.\frac{d}{dt} M_X(t)\right|_{t=0} \quad \text{and} \quad \left.\frac{d^2}{dt^2} M_X(t)\right|_{t=0}.$$

These values should agree with the values of $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$ that you have obtained in part (a).

Hint 3: for a continuous random variable X, the expectation value of the random variable $g(X)$ is given by $\int_{-\infty}^{\infty} g(x)f(x)\,dx$. Here $g(X)$ is any function of X, for example, you may take $g(X) = X^2$.

Hint 4: You may find the following integral helpful:

$$\int_0^\infty e^{-z^2/2}\,dz = \sqrt{\frac{\pi}{2}}.$$

3. Recall that (Corollary 3.8 in the textbook) if two random variables X and Y are independent, then they must be uncorrelated i.e. $\text{Cov}(X, Y) = 0$. However, the converse is not true in general and this problem provides an example.

Let X be a random variable with continuous uniform distribution on the interval $[-1, 1]$, i.e. its probability density function is given by

$$f(x) = \begin{cases} 1/2, & \text{if } x \in [-1, 1], \\ 0, & \text{otherwise.} \end{cases}$$

a) Show that $\text{Cov}(X, X^2) = 0$.

b) Prove mathematically (not just argue by intuition) that X and X^2 are not independent. One way to do this is by showing that they do not satisfy the property:

$$\mathbb{P}(X \in A, X^2 \in B) = \mathbb{P}(X \in A) \cdot \mathbb{P}(X^2 \in B)$$

for all $A, B \subseteq \mathbb{R}$. You may also use other equivalent definitions of independence.